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1999 J. Phys. A: Math. Gen. 32 2027

(http://iopscience.iop.org/0305-4470/32/10/018)

J. Phys. A: Math. Gen. **32** (1999) 2027–2036. Printed in the UK PII: S0305-4470(99)97225-6

An algebraic description of generalized k-constraints

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Received 2 September 1998

Abstract. The generalized k-constrained KP hierarchy is shown to correspond to a so-called pseudo-reduction of the two-dimensional Toda lattice hierarchy, described in a free-fermion approach which is adapted to the case of two singularities in the spectral parameter range. Wronskian solutions are discussed and, in particular, soliton solutions are recovered through a $p^k + c/p = q^k + c/q$ reduction of the Toda solitons.

1. Introduction

It has become widely accepted that imposing so-called symmetry constraints on the KP hierarchy is one of the most interesting methods for obtaining integrable $(1 + 1)$-dimensional evolution equations [1–5]. Several approaches have been developed for the implementation of this procedure: one can either impose a constraint on the linear system underlying the KP hierarchy [3, 4, 6], one can impose a 'symmetry constraint' on the level of the actual solutions of the KP evolution equations [1, 2] or on the level of the tau functions giving rise to solutions [7, 8], or one might give a geometric interpretation of such constraints [9]. While all of these approaches have their own merits, we found that imposing a symmetry constraint on the tau functions themselves has two immediate advantages. First of all, one recovers the bilinear equations and determinant-type solutions for the reduced equations with remarkable ease, but it also turns out that this approach allows for an interesting and straightforward generalization of the usual constraints [8].

This generalization proceeds as follows. It is a well known fact that the entire KP hierarchy can be encoded in the following bilinear expression [10, 11]:

$$\operatorname*{Res}_{\lambda=\infty}\left[\tau\left(\boldsymbol{x}-\epsilon\left(\frac{1}{\lambda}\right)\right)\tau\left(\boldsymbol{x}'+\epsilon\left(\frac{1}{\lambda}\right)\right)\mathrm{e}^{\xi(\boldsymbol{x}-\boldsymbol{x}',\lambda)}\right]=0 \tag{1}$$

making use of a tau function $\tau(\boldsymbol{x})$ which depends on an infinite sequence of time variables $\boldsymbol{x} = (x_1 = x, x_2, x_3, \ldots)$; the 'shift' $\epsilon(\lambda^{-1})$ stands for $(\lambda^{-1}, \frac{1}{2}\lambda^{-2}, \frac{1}{3}\lambda^{-3}, \ldots)$. The argument $\xi(\boldsymbol{x}, \lambda)$ of the exponential function appearing in this expression represents the formal series $\xi(\boldsymbol{x}, \lambda) = \sum_{n=1}^{\infty} x_n\lambda^n$. The $(2 + 1)$-dimensional nonlinear partial differential equations (NLPDEs) making up the KP hierarchy can be recovered from this expression as successive coefficients in a careful expansion in terms of $\boldsymbol{x} - \boldsymbol{x}'$.

The reduction procedure we use here to obtain $(1 + 1)$-dimensional integrable evolution equations from the KP hierarchy (and which we refer to as the generalized k-constraint), consists of imposing the following condition [8] on the KP tau functions (and hence on the bilinear expression (1)):

$$\tau_{x_k} = \tau\Omega(\Phi, \Phi^*) - cx\tau \tag{2}$$

(for an arbitrary but fixed constant c). The functions Φ and Φ^* are eigenfunctions and adjoint eigenfunctions for the KP hierarchy, meaning that they satisfy the KP (adjoint) linear problem [12, 13] ($\forall\, n \geqslant 2$):

$$p_n(-\tilde{\partial})\Phi = \Phi p_{n-1}(-\tilde{\partial})(\log\tau)_x \qquad p_n(\tilde{\partial})\Phi^* = \Phi^* p_{n-1}(\tilde{\partial})(\log\tau)_x \tag{3}$$

where $p_n(\pm\tilde{\partial})$ denote the Schur polynomials, $\sum_{n=0}^{\infty} p_n(\boldsymbol{x})\lambda^n \equiv \exp\xi(\boldsymbol{x}, \lambda)$, expressed in the variables $\tilde{\partial} = (\partial_x, \frac{1}{2}\partial_{x_2}, \frac{1}{3}\partial_{x_3}, \ldots)$. The function $\Omega(\Phi, \Phi^*)$ represents the so-called eigenfunction potential associated to the pair (Φ, Φ^*), defined by the total differential [14]

$$\mathrm{d}\Omega \equiv \Phi\Phi^*\,\mathrm{d}x + (\Phi_x\Phi^* - \Phi\Phi^*_x)\,\mathrm{d}x_2 + \cdots. \tag{4}$$

In [8] it is shown that the constraint (2) can be interpreted as a symmetry constraint as it identifies two sets of symmetries for the KP hierarchy: elements ∂_{x_k} and x of the Heisenberg algebra, on the one hand, and the product of τ and the eigenfunction potential $\Omega(\Phi, \Phi^*)$, on the other hand. It is also shown that the constraint (2) gives rise to an auxiliary set of bilinear equations for τ and that it has an interpretation on the level of the usual pseudo-differential construction of the KP hierarchy as well. If one constructs [11] the KP hierarchy starting from a pseudo-differential Lax operator $L = \partial + u_2\partial^{-1} + u_3\partial^{-2} + \cdots$, using the definitions $\partial\partial^{-1} = \partial^{-1}\partial = 1$ and $\partial f(\boldsymbol{x}) = f(\boldsymbol{x})\partial + f_x(\boldsymbol{x})$, with coefficients u_n expressible in terms of the tau functions ($u_2 = \partial_x^2 \log\tau$, $u_3 = 1/2(\partial_{x_2} - \partial_x^2)\partial_x \log\tau, \ldots$) it can be shown that the symmetry constraint (2) is equivalent to the condition

$$L^k + cL^{-1} = (L^k)_+ + \Phi\partial^{-1}\Phi^* \tag{5}$$

(the operation $(P)_+$ restricting a pseudo-differential operator P to the part with non-negative powers of ∂).

In this paper, we provide an algebraic description of this particular type of reduction, as an alternative to the above pseudo-differential approach. A key observation is that the soliton solutions of the reduced systems can be obtained from the usual KP solitons by so-called 'pseudo-reductions'. The term was coined by Hirota when studying reductions of the KP or mKP hierarchies which result in polynomial dispersion relations for the soliton solutions (as opposed to the usual monomial ones). In particular, the interest in a generalized k-constraint was triggered by the fact that, at the first level $k = 1$, it yields a system closely related to the Broer–Kaup system which was studied by Hirota [15] in terms of pseudo-reductions as far back as 1985 (in fact, it was only shown much later that the Broer–Kaup system actually is a pseudo-reduction of the mKP hierarchy [16]). For our present purposes we shall need an algebraic description of the two-dimensional (2D) Toda lattice hierarchy, in terms of (charged) free-fermion operators. In this particular framework it will be explained how the concept of a pseudo-reduction immediately leads to constraints of type (2) on the KP evolutions.

2. The two-dimensional Toda lattice hierarchy

The basic fermionic description of the 2D Toda lattice runs largely parallel to that of the KP hierarchy, the main difference being however the introduction of a new set of independent variables, associated to a second (essential) singularity in the time evolutions. For an exhaustive overview of these methods the interested reader is referred to [10, 17] (or [18, 19] for a related approach). The approach is formulated in terms of (charged) free-fermion creation and annihilation operators ψ_i, ψ_j^* (for $i, j \in \mathbb{Z} + \frac{1}{2}$) satisfying the anti-commutation relations:

$$[\psi_i, \psi_j^*]_+ := \delta_{i+j,0} \qquad \text{and} \qquad [\psi_i, \psi_j]_+ = [\psi_i^*, \psi_j^*]_+ = 0. \tag{6}$$

As usual, in the standard Fock representation of this algebra (and its dual representation), cyclic vectors $|\text{vac}\rangle$ (and $\langle\text{vac}|$) are introduced:

$$\begin{aligned} \psi_i|\text{vac}\rangle &= \psi_i^*|\text{vac}\rangle = 0 \qquad \text{for} \quad i > 0 \\ \langle\text{vac}|\psi_i &= \langle\text{vac}|\psi_i^* = 0 \qquad \text{for} \quad i < 0. \end{aligned} \tag{7}$$

Expectation values are defined using the fundamental pairing $\langle\text{vac}|1|\text{vac}\rangle = 1$. In terms of the basic fermion operators, formal operators depending on some 'spectral' parameters are defined as

$$\psi(\lambda) \equiv \sum_{j\in\mathbb{Z}+\frac{1}{2}} \psi_j\, \lambda^{-j-1/2} \qquad \psi^*(\lambda) \equiv \sum_{j\in\mathbb{Z}+\frac{1}{2}} \psi_j^*\, \lambda^{-j-1/2}. \tag{8}$$

As mentioned earlier, in order to describe the 2D Toda lattice we have to introduce a time evolution for these operators which exhibits singularities at both $\lambda = \infty$ and 0, in contrast to the ordinary KP hierarchy where there is only one singularity (at $\lambda = \infty$). First, we must define $H_n \equiv \sum_{j\in\mathbb{Z}+\frac{1}{2}} \psi_{-j}\psi_{j+n}^*$ for all integers $n \neq 0$. Due to the relations (7) one has the important properties $H_{n>0}|\text{vac}\rangle = 0$ and $\langle\text{vac}|H_{n<0} = 0$, in addition to the commutation relation: $[H_n, H_m] = n\delta_{n+m,0}$. Now we can define Hamiltonians $H^+(\boldsymbol{x})$ and $H^-(\boldsymbol{y})$ associated to both singular points $\lambda = \infty$ and 0 ($\boldsymbol{y} = (y_1 = y, y_2, y_3, \ldots)$):

$$H^+(\boldsymbol{x}) \equiv \sum_{n=1}^{\infty} x_n\, H_n \qquad H^-(\boldsymbol{y}) \equiv \sum_{n=1}^{\infty} y_n\, H_{-n} \tag{9}$$

which have the following commutation property:

$$\mathrm{e}^{H^+(\boldsymbol{x})}\, \mathrm{e}^{H^-(\boldsymbol{y})} = \mathrm{e}^{H^-(\boldsymbol{y})}\, \mathrm{e}^{H^+(\boldsymbol{x})} \mathrm{e}^{\sum_{n=1}^{\infty} n\, x_n y_n}. \tag{10}$$

In terms of these Hamiltonians, the time evolutions of the operators $\psi(\lambda)$ and $\psi^*(\lambda)$ are given through the following automorphisms [10]:

$$\psi(\lambda) \to \lambda^n\, \mathrm{e}^{H^+(\boldsymbol{x})}\, \mathrm{e}^{H^-(\boldsymbol{y})} \psi(\lambda)\, \mathrm{e}^{-H^-(\boldsymbol{y})}\, \mathrm{e}^{-H^+(\boldsymbol{x})} = \lambda^n\, \mathrm{e}^{\xi(\boldsymbol{x},\lambda)+\xi(\boldsymbol{y},\lambda^{-1})} \psi(\lambda) \tag{11}$$

$$\psi^*(\lambda) \to \lambda^{-n}\, \mathrm{e}^{H^+(\boldsymbol{x})}\, \mathrm{e}^{H^-(\boldsymbol{y})} \psi^*(\lambda)\, \mathrm{e}^{-H^-(\boldsymbol{y})}\, \mathrm{e}^{-H^+(\boldsymbol{x})} = \lambda^{-n}\, \mathrm{e}^{-\xi(\boldsymbol{x},\lambda)-\xi(\boldsymbol{y},\lambda^{-1})} \psi^*(\lambda). \tag{12}$$

Tau functions $\tau_n(\boldsymbol{x}, \boldsymbol{y})$ are introduced as vacuum expectation values for certain elements $g = \exp(\sum_{i,j} a_{ij}\, \psi(p_i)\, \psi^*(q_j) + a)$ of (a suitable completion $\overline{GL}(\infty)$ of) the group $GL(\infty)$ [10, 20]. Denoting the time evolution of such an element by $g(n; \boldsymbol{x}, \boldsymbol{y})$ we have

$$\tau_n(\boldsymbol{x}, \boldsymbol{y}) \equiv \langle\text{vac}|g(n; \boldsymbol{x}, \boldsymbol{y})|\text{vac}\rangle. \tag{13}$$

These tau functions satisfy the general bilinear identity:

$$\begin{aligned} &\operatorname*{Res}_{\lambda=\infty}\left[\lambda^{n-n'}\tau_n\left(\boldsymbol{x} - \boldsymbol{\epsilon}\left(\frac{1}{\lambda}\right), \boldsymbol{y}\right) \tau_{n'}\left(\boldsymbol{x}' + \boldsymbol{\epsilon}\left(\frac{1}{\lambda}\right), \boldsymbol{y}'\right) \mathrm{e}^{\xi(\boldsymbol{x}-\boldsymbol{x}',\lambda)+\xi(\boldsymbol{y}-\boldsymbol{y}',\lambda^{-1})}\right] \\ &\qquad + \operatorname*{Res}_{\lambda=0}\left[\lambda^{n-n'}\, \tau_{n+1}(\boldsymbol{x}, \boldsymbol{y} - \boldsymbol{\epsilon}(\lambda))\tau_{n'-1}(\boldsymbol{x}', \boldsymbol{y}' + \boldsymbol{\epsilon}(\lambda))\, \mathrm{e}^{\xi(\boldsymbol{x}-\boldsymbol{x}',\lambda)+\xi(\boldsymbol{y}-\boldsymbol{y}',\lambda^{-1})}\right] = 0. \end{aligned} \tag{14}$$

The lowest-order members (i.e. in $\boldsymbol{x}$) of the hierarchy encoded in this bilinear form are easily found to be

$$D_x D_y \tau_n \cdot \tau_n = 2[\tau_n^2 - \tau_{n-1}\tau_{n+1}] \tag{15}$$

$$D_{x_2} D_y \tau_n \cdot \tau_n = 2 D_x \tau_{n-1} \cdot \tau_{n+1}. \tag{16}$$

The first equation in this list is of course the Hirota bilinear form of the 2D Toda lattice [21]:

$$(\theta_n)_{xy} = \mathrm{e}^{-\theta_{n+1}} - 2\mathrm{e}^{-\theta_n} + \mathrm{e}^{-\theta_{n-1}} \qquad \text{for} \quad \theta_n = \log \frac{\tau_n^2}{\tau_{n+1}\tau_{n-1}} \tag{17}$$

(from now on, we shall denote the tau functions $\tau_n(\boldsymbol{x}, \boldsymbol{y})$ as τ_n). Note that taking $\boldsymbol{y}' = \boldsymbol{y}$ and $n' = n$ in the bilinear identity (14) reduces it to the KP bilinear identity (1) for $\tau = \tau_n(\boldsymbol{x}, \boldsymbol{y})$, where the typical 'Toda-like' $\boldsymbol{y}$ variables are then to be regarded as mere parameters in genuine KP tau functions. In the same way, considering the case $\boldsymbol{y}' = \boldsymbol{y}$ and $n' = n-1$, one recovers the modified KP bilinear identity [10] showing that the ratios τ_{n+1}/τ_n and τ_{n-1}/τ_n are, respectively, KP eigenfunctions and adjoint eigenfunctions associated to the tau function τ_n.

The associated linear problem for the 2D Toda hierarchy can be derived in a similar fashion. Let us define the functions

$$V_\lambda(n; \boldsymbol{x}, \boldsymbol{y}) = \langle 1|\psi(\lambda)\, g(n; \boldsymbol{x}, \boldsymbol{y})|\mathrm{vac}\rangle \tag{18}$$

$$V^*_\lambda(n; \boldsymbol{x}, \boldsymbol{y}) = \langle -1|\psi^*(\lambda)\, g(n; \boldsymbol{x}, \boldsymbol{y})|\mathrm{vac}\rangle \tag{19}$$

for states $\langle 1| = \langle \mathrm{vac}|\psi^*_{1/2}$ and $\langle -1| = \langle \mathrm{vac}|\psi_{1/2}$. They allow for the following representations in terms of tau functions [10]:

$$V_\lambda(n; \boldsymbol{x}, \boldsymbol{y}) = \begin{cases} \tau_n\left(\boldsymbol{x} - \boldsymbol{\epsilon}\left(\frac{1}{\lambda}\right), \boldsymbol{y}\right) & \text{if expanded around } \lambda = \infty \\ \tau_{n+1}(\boldsymbol{x}, \boldsymbol{y} - \boldsymbol{\epsilon}(\lambda)) & \text{if expanded around } \lambda = 0 \end{cases} \tag{20}$$

$$V^*_\lambda(n; \boldsymbol{x}, \boldsymbol{y}) = \begin{cases} \tau_n\left(\boldsymbol{x} + \boldsymbol{\epsilon}\left(\frac{1}{\lambda}\right), \boldsymbol{y}\right) & \text{if expanded around } \lambda = \infty \\ \tau_{n-1}(\boldsymbol{x}, \boldsymbol{y} + \boldsymbol{\epsilon}(\lambda)) & \text{if expanded around } \lambda = 0 \end{cases} \tag{21}$$

and give rise to wavefunctions and adjoint wavefunctions for the Toda hierarchy:

$$\psi_\lambda(n; \boldsymbol{x}, \boldsymbol{y}) = V_\lambda(n; \boldsymbol{x}, \boldsymbol{y})\, \lambda^n \mathrm{e}^{\xi(\boldsymbol{x},\lambda)+\xi(\boldsymbol{y},\lambda^{-1})}/\tau_n(\boldsymbol{x}, \boldsymbol{y}) \tag{22}$$

$$\psi^*_\lambda(n; \boldsymbol{x}, \boldsymbol{y}) = V^*_\lambda(n; \boldsymbol{x}, \boldsymbol{y})\, \lambda^{-n} \mathrm{e}^{-\xi(\boldsymbol{x},\lambda)-\xi(\boldsymbol{y},\lambda^{-1})}/\tau_n(\boldsymbol{x}, \boldsymbol{y}). \tag{23}$$

These wave and adjoint wavefunctions satisfy the Toda linear problem which can be obtained under the form of 'bilinear identities' as well. More precisely, it is possible to derive two different identities: the first one choosing λ, μ and p to lie in a neighbourhood of ∞

$$\sum_{s=\lambda,\mu,p} \operatorname*{Res}_{k=s} \left[\frac{V_k(n; \boldsymbol{x}, \boldsymbol{y})\, V^*_k(n; \boldsymbol{x} - \boldsymbol{\epsilon}(1/p) - \boldsymbol{\epsilon}(1/\lambda) - \boldsymbol{\epsilon}(1/\mu), \boldsymbol{y})}{(k-p)(k-\mu)(k-\lambda)} \right] = 0 \tag{24}$$

and the second one in the case where λ, μ and p lie in the neighbourhood of 0,

$$\sum_{s=\lambda,\mu,p} \operatorname*{Res}_{k=s} \left[\frac{V_k(n; \boldsymbol{x}, \boldsymbol{y})\, V^*_k(n+3; \boldsymbol{x}, \boldsymbol{y} - \boldsymbol{\epsilon}(p) - \boldsymbol{\epsilon}(\lambda) - \boldsymbol{\epsilon}(\mu))}{(k-p)(k-\mu)(k-\lambda)} \right] = 0. \tag{25}$$

When expressed in terms of wavefunctions and denoting $\psi_p(n; \boldsymbol{x}, \boldsymbol{y})$ as $\phi_n(\boldsymbol{x}, \boldsymbol{y})$, the identity (24):

$$\begin{aligned} \lambda\, \phi_n\left(\boldsymbol{x} - \boldsymbol{\epsilon}\left(\frac{1}{\lambda}\right), \boldsymbol{y}\right) \psi_\lambda(n; \boldsymbol{x}, \boldsymbol{y}) = \mu \Bigg[& \psi_\lambda(n; \boldsymbol{x}, \boldsymbol{y})\, \phi_n\left(\boldsymbol{x} - \boldsymbol{\epsilon}\left(\frac{1}{\mu}\right), \boldsymbol{y}\right) \\ & - \phi_n(\boldsymbol{x}, \boldsymbol{y})\, \psi_\lambda\left(n; \boldsymbol{x} - \boldsymbol{\epsilon}\left(\frac{1}{\mu}\right), \boldsymbol{y}\right) \Bigg] \end{aligned} \tag{26}$$

is recognized as a fundamental identity which holds for the eigenfunctions of the KP hierarchy in general [22]. Note that the renaming $\psi_p(n; \boldsymbol{x}, \boldsymbol{y}) \leftrightarrow \phi_n(\boldsymbol{x}, \boldsymbol{y})$ in formula (26) is not a completely innocent one: it stresses that the spectral parameter p is of no relevance in the equation as it only appears as an internal parameter in some special solutions (i.e. in the wavefunctions $\psi_p(n;\boldsymbol{x},\boldsymbol{y})$). Moreover, upon inspection, the equation actually turns out to be non-singular in the limit $p \to 0$: taking the consecutive limits $\mu \to \infty$, $p \to 0$ yields (due to relations (20) and (22))

$$\lambda \frac{\tau_{n+1}\big(\boldsymbol{x}-\boldsymbol{\epsilon}(1/\lambda),\boldsymbol{y}\big)}{\tau_n\big(\boldsymbol{x}-\boldsymbol{\epsilon}(1/\lambda),\boldsymbol{y}\big)} + \left(\frac{\tau_{n+1}}{\tau_n}\right)_x = \left(\frac{\tau_{n+1}}{\tau_n}\right)[\log\psi_\lambda(n;\boldsymbol{x},\boldsymbol{y})]_x \tag{27}$$

actually stating again that the ratio τ_{n+1}/τ_n satisfies the KP linear problem (3).

The second identity (25) can be expressed as

$$\begin{aligned}\phi_n(\boldsymbol{x},\boldsymbol{y})\,\psi_\lambda(n+1;\boldsymbol{x},\boldsymbol{y}-\boldsymbol{\epsilon}(\mu)) - \psi_\lambda(n;\boldsymbol{x},\boldsymbol{y})\,\phi_{n+1}(\boldsymbol{x},\boldsymbol{y}-\boldsymbol{\epsilon}(\mu))\\ +\psi_\lambda(n;\boldsymbol{x},\boldsymbol{y})\,\phi_{n+1}(\boldsymbol{x},\boldsymbol{y}-\boldsymbol{\epsilon}(\lambda)) = 0\end{aligned} \tag{28}$$

which encodes the Toda linear equations for the $\boldsymbol{y}$ variables. For example, taking the limit $\mu \to 0$ and shifting $n \to n-1$ in this identity, we obtain (at first order in λ) the y evolution of $\phi_n(\boldsymbol{x},\boldsymbol{y})$:

$$(\phi_n)_y = \mathrm{e}^{-\theta_n}\,\phi_{n-1} \tag{29}$$

with θ_n as in formula (17). On the other hand, if we express formula (27) completely in terms of $\phi_n = \psi_\lambda(n;\boldsymbol{x},\boldsymbol{y})$, we find (once again on account of relations (20) (first equation) and (22)) a first-order linear equation for the x evolution of the eigenfunctions ϕ_n:

$$(\phi_n)_x = v_n\,\phi_n + \phi_{n+1} \qquad \text{for} \quad v_n = \left(\log\frac{\tau_{n+1}}{\tau_n}\right)_x. \tag{30}$$

Noticing the relation $(\theta_n)_x = v_{n-1} - v_n$, it is easily verified that the system (29) and (30), is the Lax pair for the 2D Toda lattice (17) [21, 23].

3. Pseudoreductions

In the present approach, reductions can be formulated as constraints on either the group elements g which are used to define tau functions, or on their generating algebra. Here we adopt the former approach and impose the following condition on $g \in \overline{GL}(\infty)$:

$$\left(\frac{1}{c}H_k + H_{-1}\right) g \equiv g\left(\frac{1}{c}H_k + H_{-1}\right) \tag{31}$$

for some $k > 0$ and for a fixed constant c. The resulting constraint on the tau functions of the Toda lattice is easily calculated. For this we need a special instance of formula (10) stating that ($k > 0$):

$$\left[H_k, \mathrm{e}^{\pm H^-(\boldsymbol{y})}\right]_- = \pm k y_k\,\mathrm{e}^{\pm H^-(\boldsymbol{y})}. \tag{32}$$

Because of this formula and bearing in mind that $H_{k>0}|\mathrm{vac}\rangle = 0$, the constraint (31) implies for $\tau_n = \tau(n;\boldsymbol{x},\boldsymbol{y})$ that

$$\begin{aligned}\frac{1}{c}(\tau_n)_{x_k} &= \frac{1}{c}\langle\mathrm{vac}|\mathrm{e}^{H^+(\boldsymbol{x})}H_k\,\mathrm{e}^{H^-(\boldsymbol{y})}g\,\mathrm{e}^{-H^-(\boldsymbol{y})}|\mathrm{vac}\rangle\\ &= \frac{1}{c}\langle\mathrm{vac}|\mathrm{e}^{H^+(\boldsymbol{x})}\,\mathrm{e}^{H^-(\boldsymbol{y})}\,g\,(H_k + k y_k)\,\mathrm{e}^{-H^-(\boldsymbol{y})}|\mathrm{vac}\rangle\\ &\quad -\langle\mathrm{vac}|\mathrm{e}^{H^+(\boldsymbol{x})}\,\mathrm{e}^{H^-(\boldsymbol{y})}(H_{-1}\,g - g\,H_{-1})\,\mathrm{e}^{-H^-(\boldsymbol{y})}|\mathrm{vac}\rangle\\ &= -(\tau_n)_{y_1}.\end{aligned} \tag{33}$$

Hence, the condition (31) on the group elements imposes the following pseudo-reduction on the tau functions:

$$\frac{1}{c}(\tau_n)_{x_k} + (\tau_n)_y = 0 \tag{34}$$

(the term 'pseudo' is used to distinguish this type of reduction from the ordinary (KP) k-reductions for which we have that $\tau_{x_k} = 0$ implies $\tau_{x_{2k}} = \tau_{x_{3k}} = \cdots = 0$ [11]; for pseudo-reductions there is no such implication and as such no 'higher'-order time evolutions are eliminated from the hierarchy).

The constraint (34) reduces the 2D Toda lattice (15) to an equation, defined solely in terms of 'KP time variables':

$$D_x D_{x_k} \tau_n \cdot \tau_n = 2c[\tau_{n-1}\tau_{n+1} - \tau_n^2] \quad \Longleftrightarrow \quad (\log \tau_n)_{x,x_k} = qr - c \tag{35}$$

upon introduction of the functions $q = \sqrt{c}\tau_{n+1}/\tau_n$ and $r = \sqrt{c}\tau_{n-1}/\tau_n$. If, from this point on, we regard the 'higher'-order time variables $y_{n>1}$ as mere parameters in the tau functions—the above equation will act as a constraint on the remaining KP evolutions along the $\boldsymbol{x}$ variables. Note that it was argued in the previous section that the functions q and r as defined above, are (respectively) eigenfunctions and adjoint eigenfunctions for the KP linear problem. Consequently, referring to the defining property of an eigenfunction potential (4) we can look upon the product qr in the above expression as the x derivative of the eigenfunction potential $\Omega(q, r)$ associated with q and r. Expression (35) is then equivalent to

$$\Omega(q, r) = (\log \tau_n)_{x_k} + cx + F(\boldsymbol{x}, \boldsymbol{y}) \qquad \text{with} \quad F(\boldsymbol{x}, \boldsymbol{y})_x = 0. \tag{36}$$

We shall now prove that this function $F(\boldsymbol{x}, \boldsymbol{y})$ does not depend on the KP variables $\boldsymbol{x}$ and hence, when it comes to an interpretation in terms of the KP hierarchy, that it may be regarded as a mere constant of integration to be included in the potential $\Omega(q, r)$. In that case we will have shown that the pseudo-reduction (34) performed on the 2D Toda lattice is equivalent to imposing a generalized k-constraint on the $\boldsymbol{x}$-evolutions for the tau functions and, for that matter, on the entire KP hierarchy:

$$(\tau_n)_{x_k} = \tau_n \Omega(q, r) - cx\tau_n. \tag{37}$$

The essential ingredient in the proof is a general property of eigenfunction potentials which governs their behaviour under shifts of the $\boldsymbol{x}$ variables, i.e.

$$[\Omega(\Phi, \Phi^*)]\left(\boldsymbol{x} - \boldsymbol{\epsilon}\left(\frac{1}{\lambda}\right)\right) - \Omega(\Phi, \Phi^*) = -\frac{1}{\lambda}\Phi\Phi^*\left(\boldsymbol{x} - \boldsymbol{\epsilon}\left(\frac{1}{\lambda}\right)\right) \tag{38}$$

(see [8, 22] or [24] for rather different proofs of this property).

Now, using the linear equation (29) for a wavefunction $\psi_\lambda(n) = \psi_\lambda(n; \boldsymbol{x}, \boldsymbol{y})$ expanded around $\lambda = \infty$, we easily obtain that

$$(\log \psi_\lambda(n))_y = \mathrm{e}^{-\theta_n} \frac{\psi_\lambda(n-1)}{\psi_\lambda(n)} = \frac{1}{\lambda} \frac{\tau_{n+1}}{\tau_n} \frac{\tau_{n-1}\big(\boldsymbol{x} - \boldsymbol{\epsilon}(1/\lambda)\big)}{\tau_n\big(\boldsymbol{x} - \boldsymbol{\epsilon}(1/\lambda)\big)} \tag{39}$$

or, remembering the definitions of q and r:

$$q\,r\left(\boldsymbol{x} - \boldsymbol{\epsilon}\left(\frac{1}{\lambda}\right)\right) = c\lambda(\log \psi_\lambda(n))_y = c + c\lambda\left(\log \frac{\tau_n\big(\boldsymbol{x} - \boldsymbol{\epsilon}(1/\lambda)\big)}{\tau_n}\right)_y. \tag{40}$$

Performing the pseudo-reduction (34) on this last equation yields

$$-\frac{1}{\lambda}qr\left(\boldsymbol{x} - \boldsymbol{\epsilon}\left(\frac{1}{\lambda}\right)\right) = \left[\log \tau_n\left(\boldsymbol{x} - \boldsymbol{\epsilon}\left(\frac{1}{\lambda}\right)\right)\right]_{x_k} - (\log \tau_n)_{x_k} - \frac{c}{\lambda} \tag{41}$$

which, in view of formula (38), should be compared with the difference

$$[\Omega(q, r)]\left(\boldsymbol{x} - \boldsymbol{\epsilon}\left(\frac{1}{\lambda}\right)\right) - \Omega(q, r)$$

calculated from equation (36). It then follows that

$$F\left(\boldsymbol{x} - \epsilon\left(\frac{1}{\lambda}\right), \boldsymbol{y}\right) = F(\boldsymbol{x}, \boldsymbol{y}) \tag{42}$$

in other words, that it is indeed a constant with respect to the $\boldsymbol{x}$ evolutions.

The effect of the pseudo-reduction (34) on a wavefunction $\psi_\lambda(n; \boldsymbol{x}, \boldsymbol{y})$ (cf formulae (19) and (22)) is easily calculated as well:

$$\begin{aligned}\left(\frac{1}{c}\partial_{x_k} + \partial_y\right)\psi_\lambda(n; \boldsymbol{x}, \boldsymbol{y}) &= \left(\frac{1}{c}\partial_{x_k} + \partial_y\right)\langle \mathrm{vac}|\mathrm{e}^{H^+(\boldsymbol{x})}\,\mathrm{e}^{H^-(\boldsymbol{y})}\,\psi(\lambda)\,g\mathrm{e}^{-H^-(\boldsymbol{y})}|\mathrm{vac}\rangle \\ &= \frac{1}{c}\left(\lambda^k + \frac{c}{\lambda}\right)\psi_\lambda(n; \boldsymbol{x}, \boldsymbol{y})\end{aligned} \tag{43}$$

(the calculation runs along the same lines as (33)). Hence, the y part (29) of the Lax pair for the Toda lattice is changed accordingly:

$$(\phi_n)_{x_k} - \eta\phi_n + c\,\mathrm{e}^{-\theta_n}\phi_{n-1} = 0 \tag{44}$$

for some spectral parameter η in what is now, together with (30), a Lax pair for the lattice equation (35). However, if we wish to obtain a 'Lax system' for the constrained KP hierarchy, we obviously need to derive a constraint on the KP linear problem which is expressed at a single lattice site (and therefore is effectively independent of n), instead of at two sites as is the case in (44). As a matter of fact, such an equation is easily obtained by applying the operator $(\partial_x - v_n)$ to equation (44),

$$(\phi_n)_{x,x_k} - v_n(\phi_n)_{x_k} - \eta[(\phi_n)_x - v_n\phi_n] + c\,\mathrm{e}^{-\theta_n}\phi_n = 0. \tag{45}$$

Furthermore, performing the pseudo-reduction (34) on the higher-order member of the Toda lattice (16), we find

$$(\log\tau_n)_{x_2,x_k} = c[(\tau_{n+1})_x\tau_{n-1} - \tau_{n+1}(\tau_{n-1})_x]\tau_n^{-2} \tag{46}$$

which, together with equation (35), leads to the identification

$$\begin{aligned} c\,\mathrm{e}^{-\theta_n} &= (\log\tau_n)_{x,x_k} + c \\ v_n &= \frac{1}{2}\frac{(\log\tau_n)_{x_2,x_k} + (\log\tau_n)_{2x,x_k}}{(\log\tau_n)_{x,x_k} + c}. \end{aligned} \tag{47}$$

Hence, equation (45) acts as a genuine auxiliary linear equation for the system (3) and thus provides the Lax description of the constrained KP hierarchy.

To conclude, let us look at some particular solutions for the constrained KP equations. Thinking of soliton solutions, the corresponding tau functions are typically obtained from elements of $\overline{GL}(\infty)$ that are constructed through repeated application of the Bäcklund transformation ($p \neq q$)

$$g \to (C + \psi(p)\,\psi^*(q))\,g \tag{48}$$

starting from some trivial ('vacuum') element $g = 1$ [22]. Hence, if such a group element is required to commute with the combination $(1/c)H_k + H_{-1}$ as in (31), it is sufficient that $\psi(p)\,\psi^*(q)$ commutes with it. As we have ($\forall n$ integer)

$$[H_n, \psi(p)]_- = p^n\psi(p) \qquad \text{and} \qquad [H_n, \psi^*(q)]_- = -q^n\psi^*(q) \tag{49}$$

we find that this is the case iff

$$p^k + \frac{c}{p} = q^k + \frac{c}{q} \tag{50}$$

which is an example of the pseudo-reductions we mentioned in the introduction and which were studied by Hirota for the case $k = 1$.

Alternatively, we might look at typical Wronskian determinant solutions, made up of functions which satisfy the linear problem for a certain 'vacuum' tau function. In the case of the Toda lattice we have the vacuum $\tau_n \equiv 1$ $(\forall n)$ for which the linear system (29) and (30) turns into

$$(\varphi_n)_x = \varphi_{n+1} \qquad r(\varphi_n)_y = \varphi_{n-1}. \tag{51}$$

It is well known that there are size-N Wronskian/Casorati determinant solutions for the 2D Toda lattice (for arbitrary N) [25, 26], defined in terms of N different solutions $\varphi_n^{(i)}$, $i = 1, \ldots, N$ to the system (51)

$$\tau_n = \det\left[\varphi_{n+j-1}^{(i)}\right] = \det\left[(\varphi_n^{(i)})_{(j-1)x}\right] \qquad i, j = 1, \ldots, N. \tag{52}$$

The pseudo-reduction reduces the linear system (51) to the corresponding 'vacuum' case $(\theta_n = 0)$ of equation (44) (omitting the index n, which has now become superfluous)

$$(\varphi)_{x_k} - \eta\varphi + c\,\partial_x^{-1}(\varphi) = 0 \tag{53}$$

where we 'define' an appropriate inverse of the ∂_x operator by its action on the exponential functions which typically make up the solutions φ: $\partial_x^{-1}\exp\xi(\boldsymbol{x}, \lambda) \equiv \lambda^{-1}\exp\xi(\boldsymbol{x}, \lambda)$, i.e. in accordance with the first condition in (51), $(\varphi_{n-1})_x = \varphi_n$, for vacuum Toda wavefunctions $\lambda^n \exp\xi(\boldsymbol{x}, \lambda)\exp\xi(\boldsymbol{y}, \lambda^{-1})$. Hence, the constrained KP hierarchy will have Wronskian determinant solutions of type (52), expressed in terms of solutions $\varphi^{(i)}$ of the vacuum KP linear system $\varphi_{x_n}^{(i)} = \varphi_{nx}^{(i)}$ complemented with the constraint (53) for spectral parameters η_i. The soliton solutions we discussed above are, of course, included in this class if we consider $\varphi^{(i)} = \exp\xi(\boldsymbol{x}, p_i) + \exp\xi(\boldsymbol{x}, q_i)$ subject to condition (50). For completeness, we end by remarking that the (adjoint) eigenfunctions $q = \sqrt{c}\,\tau_{n+1}/\tau_n$ $(r = \sqrt{c}\,\tau_{n-1}/\tau_n)$ are also easily recovered. It suffices to note that, due to formula (52),

$$\tau_{n+1} = \det\left[\left(\varphi^{(i)}\right)_{jx}\right] \qquad i, j = 1, \ldots, N \tag{54}$$

whereas

$$\tau_{n-1} = \det\left[\left(\varphi^{(i)}\right)_{(j-2)x}\right] \qquad i, j = 1, \ldots, N \tag{55}$$

with the understanding that the entries in its first column (i.e. $j = 1$) are to be interpreted as $\partial_x^{-1}(\varphi^{(i)})$, in the sense explained above. These solutions for the generalized k-constrained KP hierarchy were first introduced in [8].

4. Conclusions

In this paper we trace back the origin of the generalized k-constrained KP hierarchy to a reduction of the (full) Toda lattice hierarchy. The link we establish is rooted in an algebraic (fermionic) description of the Toda hierarchy and as such provides an interesting contrast to the usual pseudo-differential approach(es) to symmetry constraints. It should be emphasized that when the generalized k-constraint is viewed as a reduction of the Toda lattice (cf formulae (31) and (34) or (50)), its limiting case $c \to 0$ will give rise to a 'standard' k-reduction on the KP-part of the hierarchy, rather than to the ordinary (scalar) k-constraint which is obtained from expressions (2) or (5) in case $c = 0$. Finally, we believe the present discussion to be rewarding also, because of the ease with which the pseudo-reduction is described, on the level of the equations, the linear systems or the solutions.

Acknowledgments

Both authors are affiliated to the Fund for Scientific Research (FWO), Flanders (Belgium): RW as a post-doctoral fellow and IL as a research assistant. IL wishes to thank Professor J Satsuma for hospitality and support during his stay at the University of Tokyo on the occasion of which the present work was initiated. RW acknowledges the support of the FWO through a mobility grant. The authors also wish to acknowledge financial support extended within the framework of the 'Interuniversitary Poles of Attraction Programme, contract no P4/08, Belgian State'.

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